

THE JORDAN-BROUWER THEOREM FOR GRAPHS

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ABSTRACT. We prove a discrete Jordan-Brouwer-Schoenflies separation theorem telling that a $(d - 1)$ -sphere H embedded in a d -sphere G defines two different connected graphs A, B in G such a way that $A \cap B = H$ and $A \cup B = G$ and such that the complementary graphs A, B are both d -balls. The graph theoretic definitions are due to Evako: the unit sphere of a vertex x of a graph $G = (V, E)$ is the graph generated by $\{y \mid (x, y) \in E\}$. Inductively, a finite simple graph is called contractible if there is a vertex x such that both its unit sphere $S(x)$ as well as the graph generated by $V \setminus \{x\}$ are contractible. Inductively, still following Evako, a d -sphere is a finite simple graph for which every unit sphere is a $(d - 1)$ -sphere and such that removing a single vertex renders the graph contractible. A d -ball B is a contractible graph for which each unit sphere $S(x)$ is either a $(d - 1)$ -sphere in which case x is called an interior point, or $S(x)$ is a $(d - 1)$ -ball in which case x is called a boundary point and such that the set δB of boundary point vertices generates a $(d - 1)$ -sphere. These inductive definitions are based on the assumption that the empty graph is the unique (-1) -sphere and that the one-point graph K_1 is the unique 0-ball and that K_1 is contractible. The theorem needs the following notion of embedding: a sphere H is embedded in a graph G if it is a subgraph of G and if any intersection with any finite set of mutually neighboring unit spheres is a sphere. A knot of co-dimension k in G is a $(d - k)$ -sphere H embedded in a d -sphere G .

1. INTRODUCTION

The Jordan-Brouwer separation theorem [21, 4] assures that the image of an injective continuous map $H \rightarrow G$ from a $(d - 1)$ -sphere H to a d -sphere G divides G into two compact connected regions A, B such that $A \cup B = G$ and $A \cap B = H$. Under some regularity assumptions, the Schoenflies theorem assures that A and B are d -balls. Hypersphere embeddings belong to knot theory, the theory of embedding spheres in

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other spheres, and more generally to manifold embedding theory [9]. While H is compact and homeomorphic to the standard sphere in \mathbf{R}^d , already a 1-dimensional Jordan curve $H \subset \mathbf{R}^2$ can be complicated, as artwork in [41] or Osgood's construction of a Jordan curve of positive area [40] illustrate. The topology and regularity of the spheres as well as the dimension assumptions matter: the result obviously does not hold surfaces G of positive genus. For codimension 2 knots H in a 3-sphere G , the complement is connected but not simply connected. Alexander [2] gave the first example of a topological embedding of S^2 into S^3 for which one domain A is simply connected while the other B is not. With more regularity of H , the Mazur-Morse-Brown theorem [36, 38, 5] assures that the complementary domains A, B are homeomorphic to Euclidean unit balls if the embedding of H is locally flat, a case which holds if H is a smooth submanifold of G diffeomorphic to a sphere. In the smooth case, all dimensions except $d = 4$ are settled: one does not know whether there are smooth embeddings of S^3 into S^4 such that one of the domains is a 4-ball homeomorphic but not diffeomorphic to the Euclidean unit ball. Related to this open Schoenflies problem is the open smooth Poincaré problem, which asks whether there are is a smooth 4-sphere homeomorphic but not diffeomorphic to the standard 4-sphere. If the smooth Poincaré conjecture turns out to be true and no exotic smooth 4-spheres exist, then also the Schoenflies conjecture would hold (a remark attributed in [6] to Friedman) as a Schoenflies counter example with an exotic 4-ball would lead to an exotic 4-sphere, a counter example to smooth Poincaré.

Even in the particular case of Jordan, various proof techniques are known. Jordan's proof in [21] which was unjustly discredited at first [24] but rehabilitated in [17]. The Schoenflies theme is introduced in [42, 43, 44, 45]. Brouwer [4] proves the higher dimensional theorem using p -dimensional "nets" defined in Euclidean space. His argument is similar to Jordan's proof for $d = 2$ using an intersection number is what we will follow here. The theorem was used by Veblen [48] to illustrate geometry he developed while writing his thesis advised by Eliakim Moore. The Jordan curve case $d = 2$ has become a test case for fully automated proof verifications. Its deepness in the case $d = 2$ can be measured by the fact that "4000 instructions to the computer generate the proof of the Jordan curve theorem" [16]. There are various proofs known of the Jordan-Brouwer theorem: it has been reduced to the Brouwer fixed point theorem [35], proven using nonstandard analysis [39] or dealt with using tools from complex analysis [10]. Alexander [1]

already used tools from algebraic topology and studied the cohomology of the complementary domains when dealing with embeddings of with finite cellular chains. In some sense, we follow here Alexander's take on the theorem, but in the language of graph theory, language formed by A.V. Evako in [19, 11] in the context of molecular spaces and digital topology. It is also influenced by discrete Morse theory [13, 14].

When translating the theorem to the discrete, one has to specify what a “sphere” and what an “embedding” of a sphere in an other sphere is in graph theory. We also need notions of “intersection numbers” of complementary spheres as well as workable notions of “homotopy deformations” of spheres within an other sphere. Once the definitions are in place, the proof can be done by induction with respect to the dimension d . Intersection numbers and the triviality of the fundamental group allow to show that the two components in the $(d - 1)$ -dimensional unit sphere of a vertex in G lifts to two components in the d -dimensional case: to prove that there are two complementary components one has to verify that the intersection number of a closed curve with the $(d - 1)$ -sphere H is even. This implies that if a curve from a point in A to B in the smaller dimensional case with intersection number 1 is complemented to become a closed curve in G , also the new connection from A to B has an odd intersection number, preventing the two regions to be the same. The discrete notions are close to the “parity functions” used by Jordan explained in [17]. Having established that the complement of H has exactly two components A, B , a homotopy deformation of H to a simplex in A or to a simplex B will establish the Schoenflies statement that A, B are d -balls. We will have to describe the homotopy on the regularized simplex level to regularizes things. The Jordan-Schoenflies theme is here used as a test bed for definitions in graph theory. Indeed, we rely on ideas from [31, 33].

Lets look at the Jordan case, an embedding of a circle in a 2-sphere. A naive version of a discrete Jordan theorem is the statement that a “simple closed curve in a discrete sphere divides the complement into two regions”. As illustrated in Figure (1), this only holds with a grain of salt. Take the octahedron graph G and a closed Hamiltonian path which visits all vertices, leaving no complement. It is even possible for any $m \geq 0$ to construct a discrete sphere G and a curve H in G such that the complement has m components as the curve can bubble off regions by coming close to itself without intersecting itself but still dividing up a disk from the rest of the sphere. As usual with failures of discrete versions of continuum results, the culprit is the definition, in this case, it is the definition of an “embedding”. The example of a

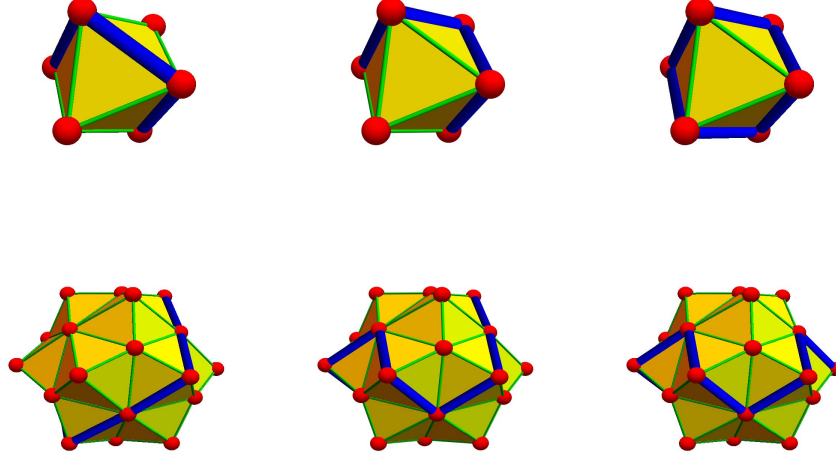


FIGURE 1. We see a homotopy deformation of an embedded C_4 graph H_1 in an octahedron G to a Hamiltonian path H_2 of length 6 in the 2-sphere G . The initial curve H_1 divides the octahedron into two complementary wheel graph domains A, B . After two steps, the deformed H_2 is no more an embedding as its closure is G . The deformations are obtained by taking a triangle t containing an edge of H and forming $H \rightarrow H \Delta t$. In the lower row, we see the situation on the simplex level, where the octahedron G has become the Catalan solid G_1 and the deformed spheres remain sphere embeddings. While our theorem will be formulated for embedded spheres in G , the proof of the Schoenflies case requires the ability to perform homotopy deformations and have a picture in which spheres remain embedded. The complementary domains A, B in the lower case are 2-balls: 9 interior vertices representing 4 triangles, 4 edges and the vertex of each pyramid at first. After the deformation there are 7 interior vertices representing the 4 triangles and 3 edges of the domain. On the simplex level the deformation steps are done using unit balls at vertices belonging to original triangles.

Hamiltonian path is more like a discrete Peano curve in the continuum as it visits all vertices but without hitting all directions or area forms of the plane. We need to make sure that also the closure of the embedding is the same curve. In the graph theoretical case, we ask that the graph generated in G by the vertex set of H is still a sphere. For curves in a 2-sphere for example we have to ask that the embedded curve features no triples of vertices forming a triangle in G . There is an other reinterpretation to make the theorem true for simple closed curves as we will see in the proof: there is a regularized picture on the simplex level, where two complementary domains always exist. We need the simplex regularization because embeddings do not play well with deformations: When making homotopy deformation steps, we in general lose the property of having an embedded sphere. Already in discrete planar geometry, where we work in the flat 2-dimensional hexagonal lattice [27] one has to invoke rather subtle definitions to get conditions which make things work. Discrete topological properties very much depend on the definitions used. It would be possible to build a homotopy deformation process which honors the embeddability, but it could be complicated. The construction of a graph product [33] provided us with an elegant resolution of the problem: we can watch the deformation of an “enhanced embedding” H_1 in G_1 , where H_1 is the graph obtained from H by taking all the complete subgraphs of H as vertices and connecting two of them if one is a subgraph of the other. It turns out that even if H is only a subgraph of G , the graph H_1 is an embedding of G_1 . This holds in particular if H is a Hamiltonian path in G , the closure of H_1 , the graph generated by the vertex set of H_1 remains geometric in G_1 . The Cartesian product [33] allows also to look at homotopy groups geometrically: a deformation of a curve in a 2-sphere G for example is now described as a geometric surface in the 3-dimensional solid cylinder $G \times L_n$, where L_n is the 1-dimensional line graph with n vertices and $(n - 1)$ edges. This is now close to the definition of homotopy of a curve using a function $F(t, s)$ in two variables so that $F(t, 0)$ is the first curve and $F(t, 1)$ the second.

This paper is not the first take on a discrete Jordan theorem as various translations of the Jordan theorem have been constructed to the discrete. They are all different from what we do here: [8] uses notions of discrete geometry, [18] looks at Jordan surfaces in the geometry of digital spaces, [12] proves a Jordan-Brouwer result in the discrete lattice Z^d , [47] extends a result of Steinhaus on a $m \times n$ checkerboard G , a minimal king-path H connecting two not rook-adjacent elements of the boundary divides G into two components. A variant of this on

a hexagonal board [15] uses such a result to prove the Brouwer fixed point theorem in two dimensions. [49] deals more generally with graphs which can have multiple connections. The result essentially establishes what we do in the special case $d = 2$. [46] looks at two theorems in $L \times L$ or $L \star L$, where \times is the usual Cartesian product and \star the tight product. In both cases, the complement of C has exactly two path components. The paper [34] deals with graphs, for which every unit sphere $S(x)$ is a Hamiltonian graph. Also here, a closed simple path C in a connected planar graph G divides the complement into exactly two components. [49] deals with graphs which can have multiple connections. On 2-spheres, it resembles the Jordan case $d = 2$ covered here as a special case.

The main theorems given here could readily be derived from the continuum by building a smooth manifold from a graph, and then use the Jordan-Brouwer-Schoenflies resp. Mazur-Morse-Brown theorem. The approach however is different as no continuum is involved: all definitions and steps are combinatorial and self-contained and could be accepted by a mathematician avoiding axioms invoking infinity. Sometimes, constructability can be a goal [3]. New is that we can prove Jordan-Brouwer-Schoenflies entirely within graph theory using a general inductive graph theoretical notion of “sphere” [11]. Papers of Jordan, Brouwer or Alexander show that the proofs in the continuum often deal with a combinatorial part only and then use an approximation argument to get the general case. As the Alexander horned sphere, the open Schoenflies conjecture or questions about triangulations related to the Hauptvermutung show, the approximation part can be difficult within topology and we don’t go into it. Our proof remains discrete but essentially follows the arguments from the continuum by defining intersection numbers and use induction with respect to dimension. The induction proof is possible because of the recursive definition of spheres and seems not have been used in the continuum, nor in discrete geometry or graph theory. But what really makes the theorem go, is to watch the story on the simplex level, where geometric graphs G remain geometric and where sub-spheres H of G can be watched as embedded spheres H_1 in G_1 . The step $G \rightarrow G_1$ can be used in the theory of triangulations because it has a “regularizing effect”. Given a triangulation G described as a graph, the new triangulation G_1 has nicer properties like that the unit sphere of a point is now a graph theoretically defined sphere.

2. DEFINITIONS

Definition 1. A subset W of the vertex set $V = V(G)$ of a graph G **generates** a subgraph (W, F) of G , where F is defined as the subset $\{(a, b) \in E \mid a \in W, b \in W\}$ of the edge set of E . The **unit sphere** $S(x)$ of a vertex x in a graph is the subgraph of G generated by all neighboring vertices of x . The **unit ball** $B(x)$ is the subgraph of G generated by the union of $\{x\}$ and the vertex set of the unit sphere $S(x)$.

If H is a subgraph of G , one can think of the graph generated by H within G as a “closure” of H within G . It is in general larger than H . For example, the closure of the line graph $H = ((a, b, c), (a, b), b, c))$ within the complete graph G with vertex set $V = \{a, b, c\}$ is equal to G .

Definition 2. Starting with the assumption that the one point graph K_1 is contractible, recursively define a finite simple graph $G = (G, V)$ to be **contractible** if it contains a vertex x such that its unit sphere $S(x)$ as well as the graph generated by $V \setminus \{x\}$ are both contractible.

Definition 3. A complete subgraph K_{k+1} of G will also be denoted **k -dimensional simplex**. If $v_k(G)$ is the set of k -dimensional simplices in G , then the **Euler characteristic** of G is defined as $\chi(G) = \sum_{k=0} (-1)^k v_k(G)$.

Examples.

- 1) The Euler characteristic of a contractible graph is always 1 as removing one vertex does not change it. One can use that $\chi(A \cap B) = \chi(A) + \chi(B) - \chi(A \cup B)$ and use inductively the assumption that unit balls as well as the spheres $S(x)$ in a reduction are both contractible.
- 2) Also by induction, using that a unit sphere of a d -sphere is $(d-1)$ -sphere, one verified that $\chi(G) = 1 + (-1)^d$ for a d -sphere. This holds also in the case $d = -1$ as the Euler characteristic of the empty graph is 0. The Euler characteristic of an octahedron for example is $6 - 12 + 8 = 2$ as there are 6 vertices, 12 edges and 8 triangles. The cube graph G is not a sphere as the unit sphere at each vertex is P_3 . Its Euler characteristic is $8 - 12 = -4$. G is a sphere with 6 holes punched in, leaving only a 1-dimensional skeleton. A 2-dimensional cube can be constructed as the boundary δB of the solid cube $B = L_2 \times L_2 \times L_2$ defined in [33].

Definition 4. Removing a vertex x from G for which $S(x)$ is contractible is called a **homotopy reduction step**. The inverse operation of performing a suspension over a contractible subgraph H of G by

adding a new vertex x and connecting x to all the vertices in H is called a **homotopy extension step**. A finite composition of reduction or extension steps is called a **homotopy deformation** of the graph G .

Remarks.

1) Since simple homotopy steps removing or adding vertices with contractible $S(x)$ do not change the Euler characteristic, it is a function on the homotopy classes [19]. If we add a vertex for which $S(x)$ is not contractible, we add a vertex with index $1 - \chi(S(x))$ which is a Poincaré-Hopf index [28]. Given a function on the vertex set giving an ordering on the build up of the graph one gets the Poincaré-Hopf theorem.

2) Examples like the Bing house or the Dunce hat show that homotopic to a one-point graph K_1 is not equivalent to contractible: some graphs might have to be expanded first before being contractible. This is relevant in Lusternik-Schnirelmann category [22].

3) The discrete notion of homotopy builds an equivalence relation on graphs in the same way homotopy does in the continuum. The problem of classifying homotopy types can not be refined as one can ask how many types there are on graphs with n vertices.

4) The discrete deformation steps were put forward by Whitehead [50] in the context of cellular complexes. The graph version is due to [19] and was simplified in [7].

5) The definition of d -spheres and d -balls is inductive. Introduced in [30, 32] we were puzzled then why this natural setup has not appeared before. But it actually has, in the context of digital topology [12] going back to [11] and we should call such spheres Evako spheres. Alexander Evako is a name shortcut for Alexander Ivashchenko who also introduced homotopy to graph theory and also as I only learned now while reviewing his work found in [20] a similar higher dimensional Gauss-Bonnet-Chern theorem [26] in graph theory.

Definition 5. The induction starts with the assumption that the empty graph is the only (-1) -sphere and that the graph K_1 is the only 0-ball. A graph is called a **d-sphere**, if all its unit spheres $S(x)$ are $(d - 1)$ spheres and if there exists a vertex $x \in V$ such that the graph generated by $V \setminus \{x\}$ is contractible. A contractible graph G is called a **d-ball**, if one can partition its vertex set V into two sets $\text{int}(V) = \{x \in V \mid S(x) \text{ is a } (d - 1) \text{ sphere}\}$ and $\delta(V) = \{x \in V \mid S(x) \text{ is a } (d - 1)\text{-ball}\}$ such that δV generates a $(d - 1)$ -sphere called δG , the **boundary of G** .

By induction, $\chi(G) = 1 + (-1)^d$ if G is a d -sphere and $\chi(G) = 1$ if G is a d -ball.

Examples.

- 1) The boundary sphere of the 0-ball K_1 is the (-1) sphere \emptyset , the empty graph.
- 2) The boundary sphere of the line graph L_n with $n > 2$ vertices, is the 0-sphere P_2 . Line graphs are 1-balls.
- 3) The boundary sphere of the wheel graph W_n with $n \geq 4$ is the circular graph C_n . The wheel graph is an example of a 2-ball and C_n is an example a 1-sphere.
- 4) The boundary sphere of the 3-ball obtained by making a suspension of a point with the octahedron is the octahedron itself.
- 5) We defined in [32] Platonic spheres as d -spheres for which all unit spheres are Platonic $(d-1)$ -spheres. This definition has been given already by Evako. The discrete Gauss-Bonnet-Chern theorem [26] easily allows a classification: all 1-dimensional spheres $C_n, n > 3$ are Platonic for $d = 1$, the Octahedron and Icosahedron are the two Platonic 2-spheres, the sixteen and six-hundred cells are the Platonic 3-spheres. As we only now realize while looking over the work of Evako, we noticed that the Gauss-Bonnet theorem [26] appears in [20]. The d -cross polytop $P_2 \star P_2 \star \cdots \star P_2$ obtained by repeating suspension operations from the 0-sphere P_2 is the unique Platonic d -sphere for $d > 3$.

Definition 6. The dimension of a graph is inductively defined as $\dim(G) = 1 + \sum_{x \in V} \dim(S(x))/v_0$, where $v_0 = |V|$ is the cardinality of the vertex set. The induction foundation is that the empty graph \emptyset has dimension 0. The dimension of a finite simple graph is a rational number.

Remarks.

- 1) This inductive dimension for graphs has appeared first in [27, 25]. It is motivated by the Menger-Uryson dimension in the continuum but it is different because with respect to the metric on a graph, the Menger-Uryson dimension is 0.
- 2) Much of graph theory literature ignores the Whitney simplex structure and treat graphs as one dimensional simplicial complexes. The inductive dimension behaves very much like the Hausdorff dimension in the continuum, the product [33] is super additive $\dim(H \times K) \geq \dim(H) + \dim(K)$ like Hausdorff dimension of sets in Euclidean space.
- 3) There are related notions of dimension like [37], who look at the largest dimension of a complete graph and then extend the dimension using the usual Cartesian product. This is not equivalent to the dimension given above.

Examples.

- 1) The complete graph K_n has dimension $n - 1$. 2) The dimension of the house graph obtained by gluing C_4 to K_3 along an edge is $22/15$: there are two unit spheres of dimension 0 which are the base points, two unit spheres of dimension $2/3$ corresponding to the two lower roof points and one unit sphere of dimension 1 which is the tip of the roof. 3) The expectation $d_n(p)$ of dimension on Erdoes-Renyi probability spaces $G(n, p)$ of all subgraphs in K_n for which edges are turned on with probability p can be computed explicitly. It is an explicit polynomial in p given by $d_{n+1}(p) = 1 + \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} d_k(p)$ [25].

Definition 7. A finite simple graph $G = (V, E)$ is called a **geometric graph** of dimension d if every unit sphere $S(x)$ is a $(d - 1)$ -sphere. A finite simple graph G is a **geometric graph with boundary** if every unit sphere $S(x)$ is either a $(d - 1)$ sphere or a $(d - 1)$ -ball. The subset of the vertex set V , in which $S(x)$ is a ball generates the **boundary graph** of G . We denote it by δG and assume it to be geometric of dimension $(d - 1)$.

Examples.

- 1) By definition, d -balls are geometric graphs of dimension d and d -spheres are geometric graphs of dimension d . 2) For every smooth d -manifold one can look at triangulations which are geometric d -graphs. The class of triangulations is much larger.

Remarks.

- 1) Geometric graphs play the role of manifolds. By embedding each discrete unit ball $B(x)$ in an Euclidean space and patching these charts together one can from every geometric graph G generate a smooth compact manifold M . Similarly, if G is a geometric graph with boundary, one can “fill it up” to generate from it a compact manifold with boundary. 2) The just mentioned obvious functor from geometric graphs to manifolds is analogue to the construction of manifolds from simplicial complexes. We don’t want to use this functor for proofs and remain in the category of graphs. One reason is that many computer algebra systems have the category of graphs built in as a fundamental data structure. An other reason is that we want to explore notions in graph theory and stay combinatorial. 3) Graph theory avoids also the rather difficult notion of triangularization. Many triangularizations are not geometric. In topology, one would for example consider the tetrahedron graph K_4 as a triangulation of the 2-sphere. But K_4 is not a sphere because unit spheres are

K_3 which are not spheres etc. And K_4 is also not a ball. While it is contractible, it coincides with its boundary as it does not have interior points. Free after Euclid one could say that K_{d+1} is a **d -dimensional point**, as it has no d -dimensional parts.

4) Every graph defines a simplicial complex, which is sometimes called the Whitney complex, but graphs are a different category than simplicial complexes. Algebraically, $x + y + z + xy + yz + xz$ is a simplicial complex which is not a graph as it does not contain the triangle simplex. The graph completion K_3 described by $x + y + z + xy + yz + zx + xyz$ however, is a graph.

Definition 8. A geometric graph of dimension d is called **orientable** if one can assign a permutation to each of its d -dimensional simplices in such a way that one has compatibility of the induced permutations on the intersections of neighboring simplices. For an orientable graph, there is a constant non-zero d -form f , called **volume form**. It satisfies $df = 0$ for the exterior derivative d but which can not be written as dh .

Remarks.

- 1) A connected orientable d -dimensional geometric graph has a 1-dimensional cohomology group $H^d(G)$. This is a special case of Poincaré duality, assuring an isomorphism of $H^{n-d}(G)$ with $H^n(G)$, which holds for all geometric graphs.
- 2) For geometric graphs, an orientation induces an orientation on the boundary. Stokes theorem for geometric graphs with boundary is $\int_G df = \int_{\partial G} f$ [29], as it is the definition on each simplex.

Examples.

- 1) All d -spheres with $d \geq 1$ are examples of orientable graphs.
- 2) If a d -sphere G has the property that antipodal points have distance at least 4, then the antipodal identification map T factors out a geometric graph G/T , we get a discrete projective space P^d . For even dimensions d , this geometric graph is not orientable.
- 3) The cylinder $C_n \times L_m$, with $n \geq 4, m \geq 2$ is orientable. One can get a sphere, a projective plane, a Klein bottle or a torus from identifications of the boundary of $L_n \times L_m$ in the same way as in the continuum. For example, the graph $L_3 \times L_3$ is obtained by taking the 25 polynomial monoid entries of $(a + ab + b + bc + c)(u + uv + v + vw + w)$ as vertices and connecting two if one divides the other.

The following definition of the graph product has been given in [33]:

Definition 9. A graph $G = (V, E)$ with vertex set $V = \{x_1, \dots, x_n\}$ defines a polynomial $f_G(x_1, \dots, x_n) = \sum_x x$, where $x = x_1^{k_1} \dots x_n^{k_n}$

with $k_i \in \{0, 1\}$ represents a complete subgraph of G . The polynomial defines the graph G_1 with vertex set $V = \{x \mid \text{simplex}\}$ and edge set $E = \{(x, y) \mid x|y \text{ or } y|x\}$, where $x|y$ means x divides y , geometrically meaning that x is a sub-simplex of y . Given two graphs H, K , define its graph product $H \times K = G(f_H \cdot f_K)$. The graph $G_1 = G \times K_1$ is called the **enhanced graph** obtained from G .

Examples.

- 1) For $G = C_4$ we have $f_G = x + xy + y + yz + z + zw + w + wx$ and $G_1 = C_8$.
- 2) For $G = K_3$, we have $f_G = x + y + z + xy + yz + zx + xyz$ and $G_1 = W_6$.
- 3) For a graph without triangles, G_1 is homeomorphic to G in the classical sense.

Remarks.

- 1) The graph G_1 has as vertices the complete subgraphs of G . Two simplices are connected if and only one is contained in the other. If G is geometric, then G_1 is geometric. For example, if G is the octahedron graph with $v_0 = 6$ vertices, $v_1 = 12$ edges and $v_2 = 8$ triangles, then G_1 is the graph belonging to the Catalan solid with $v_0 + v_1 + v_2 = 26$ vertices and which has triangular faces. Also G_1 is a 2-sphere.
- 2) In full generality, the graph G_1 is homotopic to G and has therefore the same cohomology. The unit balls of G_1 form a weak Čech cover in the sense that the nerve graph of the cover is the old graph G and two elements in the cover are linked, if their intersection is a $d - 1$ dimensional graph. To get from G_1 to G , successively shrink each unit ball of original vertices analogue to a Vietoris-Begle theorem. If G is geometric, it is possible to modify the cover to have it homeomorphic in the sense of [31] so that if G is geometric then G and G_1 are homeomorphic. As the dimension of G_1 can be slightly larger in general, the property that G_1 and G are homeomorphic for all general finite simple graphs does not hold.

Examples.

- 1) A Hamiltonian path H in the icosahedron G , a 2-sphere, does not leave any room for complementary domains. However, the graph H_1 in G_1 divides G_1 into two regions. The graph G_1 by the way is the disdyakis triacontahedron, a Catalan solid with 62 vertices.
- 2) Let G be the octahedron, a 2-sphere with 6 vertices. Assume a, b, c, d are the vertices of the equator sphere C_4 and that n, p are the north and south pole. Define the finite simple curve a, b, c, d, s, a of length 5. It is

a circle H in G , but it is not embedded. In this case, the complement of H has only one region $A = \{n\}$. The Jordan-Brouwer theorem is false. But its only in this picture. If we look at the embedding of H_1 in G_1 , then this is an embedding which divides the Catalan solid G_1 into two regions A_1, B_1 .

Definition 10. A graph $H = (W, F)$ is a subgraph of $G = (V, E)$ if $W \subset V$ and $F \subset E$. Let H, G be geometric graphs. A graph H is **embedded** in an other graph G if H is a subgraph such that for any collection of unit spheres $S(x_j)$ in G , where x_j is in the vertex set of some K_k , the intersection $H \cap \bigcap_{j=1}^k S(x_j)$ is a sphere.

Examples.

- 1) A 0-sphere H is embedded in a geometric graph if the two vertices are not adjacent. A 1-sphere H is embedded if two vertices of H are connected in G if and only if they are connected in H .
- 2) A graph C_k can only be immersed naturally in C_n if n divides k and $n \geq 4$. It is a curve winding k/n times around C_n . For example, C_{15} can be immersed in C_5 and described by the homomorphism algebraically described by $x \in Z_{15} \rightarrow x \in Z_5$ if C_n is identified with the additive group Z_n . In this algebraic setting dealing with the fundamental group it is better to look at the graph homomorphism rather than the physical image of the homomorphism.
- 3) If H is embedded in G , then also H_1 is an embedding of G_1 . But H_1 is an embedding in G_1 even if H is only a subgraph of G . See Figure (1).

The following definition places the sphere embedding problem into the larger context of knot theory:

Definition 11. A **knot of co-dimension k** is an embedding of a $(d - k)$ -sphere H in a d -sphere G .

Remarks.

- 1) A knot can be called **trivial** if it is homeomorphic to the $(d - k)$ -cross polytop embedded in the d -cross polytop in the sense of [31].
- 2) As we don't yet know whether there are graphs homeomorphic to spheres which are not spheres or whether there are graphs homeomorphic to balls which are not balls.
- 3) The Jordan-Brouwer-Schoenflies theorem can not be stated in the form that a $(d - 1)$ -sphere in a d sphere is trivial.

Definition 12. A **closed curve** in a graph is a sequence of vertices x_j with $(x_j, x_{j+1}) \in E$ and $x_n = x_0$. A **simple curve** in the graph is the image of an injective graph homomorphism $L_n \rightarrow G$, where L_n is

the line graph. A **simple closed curve** is the image of an injective homomorphism $C_n \rightarrow G$ with $n \geq 3$. It is an embedding of the circle if the image generates a circle. In general, a simple closed curve is not an embedding of a circular graph.

Example.

1) A Hamiltonian path is a simple closed curve in a graph G which visits all vertices exactly once. Such a path is not an embedding if G has dimension larger than 1 as illustrated in Figure (1).

2) While we mainly deal with geometric graphs, graphs for which all unit spheres are spheres, the notion of a simple closed curve or an embedding can be generalized for any pair of finite simple graphs H, G : if H is a subgraph of G , then there is an injective graph homomorphism from H to G . If the intersection of an intersection of finitely many neighboring unit spheres with H is a sphere, we speak of an embedding.

Definition 13. An embedding of a graph H in G **separates** G into two graphs A, B if $A \cap B = H$, $A \cup B = G$ and $A \setminus H, B \setminus H$ are disjoint nonempty graphs. The two graphs A, B are called **complementary subgraphs** of the embedding H in G .

Examples.

1) The empty graph separates any two connectivity components of a graph.

2) By definition, if a graph G is k -connected but $(k+1)$ -disconnected, there is a graph H consisting of k vertices such that H separates G .

3) For $G = K_n$, there is no subgraph H which separates G .

4) The join of the 1-sphere C_4 with the 0-sphere P_2 is a 2-sphere, the disdyakis dodecahedron, a Catalan solid which by the way is $G \times K_1$, where G is the octahedron. See Figure (1).

The C_4 subgraph embedded as the equator in the octahedron G separates G into two wheel graphs A, B .

Remarks.

1) A knot H of co-dimension 2 in a 3-sphere G is a closed simple curve embedded in G . Classical knots in R^3 can be realized in graph theory as knots in d -spheres, so that the later embedding is the same as in the discrete version. The combinatorial problem is not quite equivalent however as it allows refined questions like how many different knot types there are in a given 3-sphere G and how many topological invariants are needed in a given 3-sphere to characterize any homotopy type or a knot of a given co-dimension. .

2) An embedded curve has some “smoothness”. In a 2-sphere for example, it intersects every triangle in maximally 2 edges. An extreme

case is a Hamiltonian graph H inside G which by definition generates the entire graph. A Hamiltonian path H which is a subgraph of a higher dimensional graph plays the role of a space-filling Peano curve in the continuum, a continuous surjective map from $[0, 1]$ to the 2-manifold M . For a simple curve H which is not an embedding in G , the complement of H can therefore be empty.

Definition 14. A **simple homotopy deformation** of a $(d-1)$ -sphere H in a d -sphere G is obtained by taking a d -simplex x in G which contains a non-empty set Y of $(d-1)$ -simplices of H and replacing these simplices with Y' , the set of $(d-1)$ -simplices in x which are in the complement Y .

Examples.

- 1) If H is a simple curve in a 2-sphere G and x is a triangle containing a single edge e of H , replace e with the two other edges of the triangle. This stretches the curve a bit. The reverse operation produces a “shortcut” between two vertices (a, b) visited by the curve initially as (a, c, b) .
- 2) If H is a 2-dimensional graph, then a homotopy step is done by replacing a triangle in the tetrahedron with the 3 other triangles of a tetrahedron.

Remarks.

- 1) It is allowed to replace an entire d -simplex with the empty graph to allow a smaller dimensional sphere to be deformed to the empty graph. This is not different in the continuum, where we deform curves to a point.
- 2) A homotopy step does not honor embeddings in general. However, it preserves the class of simple curves with the empty curve included.
- 3) We have in the past included a second homotopy deformation which removes or adds backtracking parts (a, b, a) . Since is only needed if one looks at homotopy deformations of general curves, we don't use it. Homotopy groups must be dealt with using graph morphisms, rather than graphs. The backtracking deformation steps would throw us from the class of simple curves.

Definition 15. We say that a k -sphere H is **trivial** in a d -sphere G if there is a sequence of simple homotopy deformations of H which deforms H to the empty graph. If every 1-sphere is trivial in G , then G is called **simply connected**. If every k -sphere is trivial in G , we say the k 'th homotopy class is trivial.

Remarks.

- 1) The set of simple closed curves is not a group, as adding a curve to itself would cross the same point twice. Similarly, the set of simple k -spheres is not a group. In order to define the fundamental group, one has to look at graph homomorphism and not at the images. This is completely analogue to the continuum, where one looks at continuous maps from T^1 to G .
- 2) Unlike in the continuum, where the zero'th homotopy set $\pi_0(G)$ is usually not provided with a group structure, but $\pi_0(G)$ can have a group structure. It is defined as the commutative group of subsets of V with the symmetric difference Δ as addition, modulo the subgroup generated by sets $\{\{a, b\} \mid (a, b) \in E\}$. It is of course $Z_0^{b_0}$ where $b_0 = \dim(H^0(G))$ is the number of connectivity components.
- 3) The Hurewicz homomorphism $\pi_0(G) \rightarrow H^0(G)$ maps a subset A of V to a locally constant function obtained by applying the heat flow $e^{-L_0 t}$ on the characteristic function $1_A(x)$ which is 1 on A and 0 else, playing the role of a 0-current = generalized function in the continuum.
- 4) Also the Hurewicz homomorphism $\pi_1(G) \rightarrow H^1(G)$ is explicit by applying the heat flow $e^{-L_1 t}$ on the function on edges telling how many times the curve has passed in a positive way through the edge.

Examples.

- 1) Every simple curve in a 2-sphere is trivial if it can be deformed to the empty graph. This general fact for d -spheres is easy to prove in the discrete setup because by definition, a d -sphere becomes contractible after removing one vertex. The contraction of this punctured sphere to a point allows a rather explicit deformation of the curve to the empty graph.
- 2) The deformation works also for 0-spheres. In a connected graph, any embedded 0 sphere can be homotopically deformed to the empty graph. So, a graph is connected if and only if every 0-sphere in G is trivial.

Definition 16. Fix a geometric d -dimensional graph G . Let $\pi_k(G)$ denote the union of all graph homomorphisms from a graph in the set $\{C_k, k \geq 3\}$ to G . Any such homomorphism $\phi : H \rightarrow G$ defines the **homomorphism graph**, for which the vertices are the union of the vertices of H and G and for which the edge set is the union of the edges in C and G together with all pairs $(a, \phi(a))$. Two such homomorphisms are called homotopic, if the corresponding homomorphism

graphs are homotopic. The homotopy classes $\pi_1(G)$ define the **fundamental group** of G . The 0-element in the group is the homotopy class of a map from the empty graph to G . The addition of two maps $C_k \rightarrow G, C_l \rightarrow G$ is a map from $C_{k+l} \rightarrow G$ obtained in the usual way by first deforming each map so that $\phi_i(0) = x_0$ is a fixed vertex x_0 , then define $\phi(t) = \phi_1(t)$ for $t \leq k$ and then $\phi(t) = \phi_2(t-k)$ for $k \leq t \leq k+l$.

Remarks.

- 1) If one would realize the graph in an Euclidean space and see it as a triangularization of a manifold, then the fundamental groups of G and M were the same. The groups work also in higher dimensions. As we have to cut up a sphere at the equator to build the addition in the higher homotopy groups, it would actually be better to define the addition in the enhanced picture and look at maps $H_1 \rightarrow G_1$ where H_1 is the enhanced graph of the k -sphere H and G_1 the enhanced version of the graph G . A deformation of a graph H to a graph K is then geometrically traced as a surface.
- 2) As a single basic homotopy extension step $C_k \rightarrow G$ to $C_{k+1} \rightarrow G$ keeps the map in the same group element of $\pi_1(G)$, the verification that the group operation is well defined is immediate.

There are various generalized notions of “geometric graphs”, mirroring the definitions from the continuum. We mention them in the next definition, as we still explore discrete versions of questions related to Schoenflies problem in the continuum. The main question is whether there are discrete versions of exotic spheres, spheres which are homeomorphic to a d -sphere but for which unit spheres are not spheres.

Definition 17. A **homology d -sphere** is a geometric graph of dimension d which has the same homology than a d -sphere. It is a geometric graph of dimension d with Poincaré polynomial $p_G(x) = 1 + x^d$. A **homology graph** of dimension d is a graph for which every unit sphere is a homology sphere. A **pseudo geometric graph** of dimension d is a graph for which every unit sphere is a finite union of $(d-1)$ spheres. A **discrete d -variety** is defined inductively as a graph for which every unit sphere is a $(d-1)$ -variety with the induction assumption that a (-1) -variety is the empty graph.

Examples.

- 1) An example of a homology sphere can be obtained by triangulating the dodecahedron and doing identifications as in the continuum. A suspension of a homology sphere is an example of a homology graph.
- 2) A figure eight graph is an example of a pseudo geometric graph of dimension 1.

3) The cube graph or dodecahedron graph are examples of discrete 1-varieties; their unit spheres are the 0-dimensional graphs P_3 which are not 0-spheres but 0-varieties.

Definition 18. Two k -spheres H, K in a d -sphere G are called **geometric homotopic within G** if there is a geometric $(k+1)$ -dimensional graph with boundary M in the $(d+1)$ -dimensional graph $G \times L_n$ such that $M \cap (G \times \{0\}) = H \times \{0\}$ and $M \cap (G \times \{n\}) = K \times \{0\}$ and such that the boundary of M is included in the boundary of $G \times L_n$.

Remarks.

1) Given a $(d-1)$ -sphere H embedded in a d -sphere G . The deformation $H \rightarrow H' = H \Delta S(x)$ is equivalent to a homotopy deformation of the complement.

2) The above definition can also be done for more general k -spheres (where k is not necessarily $d-1$) by taking intersections of unit spheres with $(k+1)$ -sphere and performing the deformation within such a sphere.

3) Any homotopy deformation of H within G defines a deformation of the embedding of H_1 in G_1 and can be seen as a geometric homotopy deformation, a surface in $G \times L_2$. We will explore this elsewhere. Let f be the polynomial in the variables x_1, \dots, x_n representing vertices in G . The function $f(x_1, \dots, x_n)(a + ab + b)$ describes the $(d+1)$ -dimensional space $G \times L_2$ in which we want to build a surface. Let $g(y_1, \dots, y_m)$ be the function describing the surface H in G , where y_i are the vertices in H . Make the deformation at x : Define $ag(x_1, \dots, x_n) + ab[g_o(S(x))x] + b[x + g_e(S(x)) + g(x_1, \dots, x_n)]$. For example if $f_H = a + c + ac$, $f_K = a + b + c + ab + bc$, $f = u(a + b + c + ab + bc) + uv(abc + a + c) + v(a + c + ac)$.

4) In [31] we wondered what the role of 1 in the ring describing graphs could be. It could enter in reduced cohomology which is used in the Alexander duality theorem $\bar{b}_k(H) = \bar{b}_{d-k-1}(G-H)$ (going back to [23]). Here $b_k = \bar{b}_k$ if $k > 0$ and $b_k = 1 + \bar{b}_k$ if $k = 0$. In the Jordan case for example, where $b(H) = (1, 1, 0)$ and $b(G-H) = (2, 0, 0)$ one gets $\bar{b}(H) = (0, 1)$ and $\bar{b}(G-H) = (1, 0)$. When embedding a 2 sphere H in a 3-sphere G , then $b(H) = (1, 0, 1)$ and $\bar{b}(H) = (0, 0, 1)$ as well as $b(G-H) = (2, 0, 0)$ and $\bar{b}(G-H) = (1, 0, 0)$. This works for any d as $b(H) = (1, 0, \dots, 0, 1)$ and $b(G-H) = (2, 0, \dots, 0)$ by Schoenflies.

3. TOOLS

In this section, we put together three results which will be essential in the proof. The first is the triviality of the fundamental group in a sphere:

Lemma 1 (trivial fundamental group). *Every embedding of a 1-sphere H in a d -sphere G for $d > 1$ is homotopic to a point.*

Proof. Look at the sphere H_1 in G_1 . Remove a vertex x disjoint from H_1 . By definition of a d -sphere, this produces a d -ball. The curve H_1 is contained in this ball B . We can now produce a homotopy deformations at the boundary of B until H is at the boundary. Note that B does not remain a ball in general during this deformation as the boundary might not generate itself but a larger set. But B_1 remains a ball. Once the H is at the boundary switch and make homotopy deformations of H until H again in the interior of B . Continuing like that, perform alternating homotopy deformations of B and H . Because the ball B can be deformed to a point, we can deform H to the empty graph. \square

We now show that if sphere H is a subgraph of G , then the enhanced sphere H_1 is embedded. It is an important point but readily follows from the definitions:

Proposition 2. *If H is a $(d - 1)$ -sphere which is a subgraph of a d -sphere G , then H_1 is embedded in G_1 .*

Proof. We have to show that $H_1 \cap S(x_1) \cap \cdots \cap S(x_k)$ is a $(d - k - 1)$ -sphere for every k and every simplex x with vertices x_1, \dots, x_k in H_1 . Any intersections are the same whether we see H as part of G or whether H is taken alone. The reason is that any of the unit spheres $S(x_k)$ consists of simplices which either contain x_k or are contained in x_k . None of these simplices invoke anything from G . So, the statement reduces to the fact that the intersection of spheres $S(x_k)$ with x_k belonging to a simplex form a sphere. But this is true by induction. For one sphere it is the definition of a sphere. If we add an additional sphere, we drill down to a unit sphere in a lower dimensional sphere. \square

Finally, we have to look at an intersection number. At appears at first that we need a transversality condition when describing spheres K, H of complementary dimension $1, d - 1$ in a d -sphere G . While we will not require transversality, the notion helps to visualize the situation.

Definition 19. Given an embedding of a $(d - 1)$ -sphere H in a d -sphere G and a simple curve C in G . We say it C **crosses H transversely**

if for every t such that $C(t) \in H$, both $C(t-1)$ and $C(t+1)$ are not in H .

More generally:

Definition 20. Given a d -sphere G , let K be an embedded k -sphere and let H be an embedded $(d-k)$ sphere. We say K, H **intersect transversely** if H_1, K_1 intersect in a 0-dimensional geometric graph.

Remarks

1) Given two complementary spheres H, K in a sphere G . Look at the spheres H_1, K_1 in G_1 . There is always a modification of the spheres so that they are transversal. Consider for example the extreme example of two identical 1-spheres H, K in the equator of the octahedron G . The graphs H_1, K_1 are closed curves of length 8 inside the Catalan solid G_1 . Now modify the closed curves by forcing H_1 to visit the vertices in G_1 corresponding to the original 4 upper triangles and K_1 visit the vertices in G_1 corresponding to the original 4 lower triangles. The modified curves now intersect in 4 vertices.

2) Given a $(d-1)$ -sphere H embedded in a d -sphere G and given a closed curve C which is transverse to H let x_j be the finite intersection points. We need to count these intersection points. For example, if a curve is just tangent to a sphere, we have only one intersection point even so we should count it with multiplicity 2. When doing homotopy deformations, we will have such situations most of the time. As we can not avoid losing transversality when doing deformations, it is better to assign intersection numbers in full generality, also if we have no transversality.

For the following definition, we fix an orientation of the 1-sphere C which we only require to be a simple closed curve and not an embedding and we fix also an orientation on the $(d-1)$ -sphere H . Since both are spheres and so orientable, this is possible.

Definition 21. If the vertex set of C is contained in the vertex set of H , we define the **intersection number** to be zero. Otherwise, let $\{a = C(t_1), \dots, C(t_k) = b\} \subset H_1$ be a connected time interval in H_1 . Let $y = C(t_0)$ be the vertex in $G \setminus H$ just before hitting H_1 and $z = C(t_{k+1})$ the vertex just after leaving H_1 . The vertex a is contained in a d -dimensional simplex generated by the edge (y, a) in C and a simplex σ in H_1 containing a . This d -simplex has an orientation from the orientation of the edge in the curve and the simplex σ in H . If the orientation of the simplex generated by C and H at the point agrees with the orientation of the d -simplex coming from G_1 , define the

incoming intersection number to be 1. Otherwise define it to be 0. Similarly, we have an outgoing intersection number which is again either 0 or 1. The sum of all the incoming and outgoing intersection numbers is called the **intersection number** of the curve.

In other words, if the curve C touches H and bounces back after possibly staying in H for some time then the intersection number is 2 or -2 , depending on the match of orientation of this “touch down”. If the curve C passes through H , entering on and leaving on different sides, then the intersection number belonging to this “crossing” is assumed to be 1. If a curve is contained in H , then the intersection number is 0. The intersection number depends on the choice of orientations chosen on C and H , but these orientations only affect the sign. If the orientation of the curve or the $(d-1)$ -sphere is changed, the sign changes, but only by an even number.

Lemma 3. *Two homotopic curves in a sphere G with embedded $(d-1)$ sphere have the same intersection number modulo 2.*

Proof. Just check it for a single homotopy deformation step of the curve. As these steps are local, a change of the intersection number can only happen if a curve makes a touch down at H before the deformation and afterwards does no more intersect. This changes the intersection number by 2. \square

The following lemma will be used in Jordan-Brouwer and is essentially equivalent to Jordan-Brouwer:

Lemma 4. *The intersection number of a closed curve K with an embedded $(d-1)$ -sphere H in a d -sphere G is always even.*

Proof. The sphere G is simply connected, so that every curve C can be deformed to a vertex not in H . For the later, the intersection number is zero. \square

In the case of transversal intersections, the result can be illustrated geometrically. Look at the intersection of the 2-dimensional deformation surface of the curve with the d -dimensional deformation cylinder $H \times L_n$. As long as the curve never intersects H in more than two adjacent vertices, this is a 1-dimensional geometric graph, which must consist of a finite union of circular graphs.

4. THE THEOREMS

Theorem 5 (Discrete Jordan-Brouwer). *A $(d-1)$ -sphere H embedded in a d -sphere G separates G into two complementary components A, B .*

Proof. We use induction with respect to the dimension d of the graph G . For $d = 0$, the graph G is the 0-sphere and H is the empty graph S_{-1} and the complementary components A, B are both K_1 graphs which by definition are 0-balls.

To prove the theorem, we prove a stronger statement: if H is a $(d-1)$ -sphere which is a subgraph of G , then H_1 separates G_1 into two components A, B .

Take a vertex $x \in H \subset G$. When removing it, by definition of spheres, it produces the $(d-1)$ -ball $H' = G \setminus \{x\}$ and the d -ball $G' = G \setminus \{x\}$. By definition of embedding, the boundary of H' is $(d-2)$ -sphere which is a subgraph of the boundary $S(x)$ of G' .

By induction, H'_1 divides G'_1 into two connected parts A' and B' .

Let A'' be the path connected component in $G_1 \setminus H_1$ containing A' and let B'' be the path connected component in $G_1 \setminus H_1$ containing B' . The union of A'' and B'' is G if there was a third component it would have H in its boundary and so intersect $S(x)$, where by induction only two components exist.

The graphs $A = A'' \cup H_1$ and $B = B'' \cup H_1$ are d -dimensional graphs which cover G_1 and have the $(d-1)$ sphere H_1 as a common boundary.

It remains to show that the two components A'', B'' are disconnected. Assume they are not. Then there is a path C in G_1 connecting a vertex $x \in A' \subset S(x)_1$ with a vertex $y \in B' \subset S(x)_1$ such that C does not pass through H . Define an other path C' from x to y but within $S(x)$. The path C' passes through H and the sum of the two paths C, C' form a closed path whose intersection number with H is 1. This contradicts the intersection lemma (4). \square

Remarks.

1) As we know $\chi(G) = 1 + (-1)^d$ and $\chi(H) = 1 - (-1)^d$ we have $2 = \chi(G) + \chi(H) = \chi(A) + \chi(B)$ so that $\chi(A) + \chi(B) = 2$. We will of course know that $\chi(A) = \chi(B) = 1$ but $\chi(A) + \chi(B) = 2$ is for free.

2) The proof used only that G is simply connected. We don't yet know that A, B are both simply connected. At this stage, there would still be some possibility that one of them is not similarly as the Alexander horned sphere.

Examples.

1) The empty graph separates the two point graph P_2 without edges into two one point graphs. 2) The two point graph P_2 without edges

separates C_4 into two 1-balls K_2 .

- 3) The equator C_4 in an octahedron O separates O into two wheel graphs W_4 .
- 4) The unit circle $S(x) = C_5$ of a vertex x in an icosahedron separates it into a wheel graph W_5 and a 2-ball with 11 vertices.
- 5) The $(d-1)$ -dimensional cross polytop embedded in a d -dimensional cross polytop separates it into two d -balls.
- 6) More generally, a d -sphere H divides the suspension $G = S_0 \star H$ into two balls, which are both suspensions of H with a single point.

Theorem 6 (Jordan-Schoenflies). *A $(d-1)$ -sphere H embedded in a d -sphere G separates G into two complementary components A, B such that A, B are both d -balls.*

Proof. Jordan-Brouwer gives two complementary domains A, B in G . They define complementary domains A_1, B_1 in G_1 , where A_1, B_1 have a nonzero number of vertices. The proof goes in two steps. We first show that A_1, B_1 are balls: to show that A_1 is a d -ball, (the case of B_1 is analog), we only have to show that A_1 is contractible, as the boundary is by definition the sphere H . It is enough to verify that we can make a homotopy deformation step of H so that the number of vertices in A_1 gets smaller. This can be done even if the complement of H has no vertices left, like for example if H would be a Hamiltonian path where A is empty. A simple homotopy deformation of H will reduce the number of vertices in A_1 and since A_1 only has a finite number of vertices, this lead to a situation, where H_1 has been reduced to a unit sphere meaning that H has reduced to a simplex verifying that A_1 is a d -ball. The deformation step is done by taking any vertex x in A_1 which belongs to a d -simplex x in A . The step $H \rightarrow H\Delta x$ produces the step $H_1 \rightarrow H_1\Delta S(x)$ in G_1 . This finishes the verification that A_1, B_1 are d -balls. But now we go back to the original situation and note that A and A_1 originally are homeomorphic. There is a concrete cover of A_1 with d -balls such that the nerve graph is A . So, also A is contractible. \square

Remarks.

- 1) Without the enhanced picture of the embedding H_1 in G_1 , there would be a difficulty as we would have to require the embedded graph H to remain an embedding. The enhanced picture allows to watch the progress of the deformation.
- 2) As shown in the proof, one could reformulate the result by saying

that if a $(d - 1)$ -sphere H is a subgraph of a d -sphere G (not necessarily an embedding) then H_1 separates G_1 into two complementary components A_1, B_1 which are both d -balls.

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